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On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation

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Abstract

Some Ostrowski type inequalities are given for the Stieltjes integral where the integrand is absolutely continuous while the integrator is of bounded variation. The case when $|f'|$ is convex is explored. Applications for the mid-point rule and a generalised trapezoid type rule are also presented.

Keywords: Integral inequalities; Ostrowski inequality; Stieltjes integral; Quadrature rules; Mid-point rule; Trapezoidal rule; Convex functions

1. Introduction

The following result is known in the literature as *Ostrowski's inequality* [1]:

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M \tag{1.1}$$

for all $x \in (a, b)$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The above result has been naturally extended for absolutely continuous functions and Lebesgue p-norms of the derivative f' in [2–4]. They can also be obtained, in a slightly different form, as particular cases of some results established by Fink in [5] for n-time differentiable functions.

For other Ostrowski type inequalities concerning Lipschitzian and $r - H$-Hölder type functions, see [6,7].

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The cases of bounded variation functions and monotonic functions were considered in [8,9] while the case of convex functions was studied in [10].

In an effort to obtain an Ostrowski type inequality for the Stieltjes integral, which obviously contains the weighted integrals case, Dragomir established in [11] the following result:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation and $u : [a, b] \to \mathbb{R}$ a function of $r - H$-Hölder type, i.e.,*

$$|u(x) - u(y)| \le H|x - y|^r \quad \text{for any } x, y \in [a, b], \tag{1.2}$$

where $r \in (0, 1]$ and $H > 0$ are given. Then, for any $x \in [a, b]$,

$$\left|[u(b) - u(a)] f(x) - \int_a^b f(t)\,\mathrm{d}u(t)\right| \le H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right]$$

$$\le H \times \begin{cases} \left[(x-a)^r + (b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases} \tag{1.3}$$

where $\bigvee_c^d (f)$ denotes the total variation of f on the interval $[c, d]$.

The dual case was considered in [12] and can be stated as follows:

Theorem 2. *Let $u : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a function of $r - H$-Hölder type. Then*

$$\left|[u(b) - u(a)] f(x) - \int_a^b f(t)\,\mathrm{d}u(t)\right| \le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) \tag{1.4}$$

for any $x \in [a, b]$.

For other results concerning inequalities for Stieltjes integrals, see [13–16].

The aim of the present paper is to continue the study of Ostrowski type inequalities for Stieltjes integrals $\int_a^b f(t)\,\mathrm{d}u(t)$ where the function f, the *integrand*, is assumed to be absolutely continuous while the *integrator* u, is of bounded variation. Applications to the mid-point rule and for a generalised trapezoid rule are also pointed out.

2. General bounds for absolutely continuous functions

The following representation result is of interest:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ such that the Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$ exists. Then*

$$f(x)[u(b) - u(a)] - \int_a^b f(t)\,\mathrm{d}u(t) = \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,\mathrm{d}\lambda\right)\mathrm{d}u(t) \tag{2.1}$$

or, equivalently,

$$\int_a^b u(t)\,\mathrm{d}f(t) - u(b)[f(b) - f(x)] - u(a)[f(x) - f(a)]$$

$$= \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,\mathrm{d}\lambda\right)\mathrm{d}u(t) \tag{2.2}$$

for each $x \in [a, b]$.

Proof. Since f is absolutely continuous on $[a, b]$, hence, for any $x, t \in [a, b]$ with $x \neq t$, one has

$$\frac{f(x) - f(t)}{x - t} = \frac{\int_t^x f'(u)\,\mathrm{d}u}{x - t} = \int_0^1 f'[(1-\lambda)x + \lambda t]\,\mathrm{d}\lambda$$

giving the equality of interest (see also [17]):

$$f(x) = f(t) + (x - t)\int_0^1 f'[(1-\lambda)x + \lambda t]\,\mathrm{d}\lambda \tag{2.3}$$

for any $x, t \in [a, b]$.

Integrating the identity (2.3) we deduce

$$f(x)\int_a^b \mathrm{d}u(t) = \int_a^b f(t)\,\mathrm{d}u(t) + \int_a^b (x - t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,\mathrm{d}\lambda\right)\mathrm{d}u(t),$$

which is exactly the desired representation (2.1).

Now, on utilising the integration by parts formula for the Stieltjes integral in the left side of (2.1), we obtain (2.2). ■

For an absolutely continuous function $f : [a, b] \to \mathbb{R}$, let us denote by $\mu(f; x, t) := |\int_0^1 f'[\lambda t + (1-\lambda)x]\,\mathrm{d}\lambda|$, where $(t, x) \in [a, b]^2$. It is obvious that, by the Hölder inequality, we have

$$\mu(f; x, t) \leq \begin{cases} \|f'\|_{[t,x],\infty} & \text{if } f' \in L_\infty[a, b]; \\ \|f'\|_{[t,x],p} & \text{if } f' \in L_p[a, b],\ p \geq 1, \end{cases} \tag{2.4}$$

where

$$\|f'\|_{[t,x],\infty} := \sup_{\substack{u\in[t,x] \\ (u\in[x,t])}} |f'(u)|,$$

$$\|f'\|_{[t,x],p} := \left|\int_t^x |f'(u)|^p \mathrm{d}u\right|^{\frac{1}{p}}, \quad p \geq 1$$

and $t, x \in [a, b]$.

We can state now the following inequality of Ostrowski type for the Stieltjes integral that provides various bounds which complement the results in Theorems 1 and 2 from the introduction:

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function and $u : [a, b] \to \mathbb{R}$ a function of bounded variation on $[a, b]$. Then*

$$\left|[u(b) - u(a)]f(x) - \int_a^b f(t)\,\mathrm{d}u(t)\right| \leq M(x), \tag{2.5}$$

and, equivalently,

$$\left|\int_a^b u(t)\,\mathrm{d}f(t) - u(b)[f(b) - f(x)] - u(a)[f(x) - f(a)]\right| \leq M(x), \tag{2.6}$$

where $M(x) = M_1(x) + M_2(x)$ and

$$M_1(x) := \bigvee_a^x (u) \sup_{t\in[a,x]} [(x - t)\mu(f; x, t)],$$

$$M_2(x) := \bigvee_x^b (u) \sup_{t\in[x,b]} [(t - x)\mu(f; x, t)],$$

for $x \in [a, b]$.

Proof. We use the fact that, if $p, v : [c, d] \to \mathbb{R}$ are such that p is continuous and v is of bounded variation, then the Stieltjes integral $\int_c^d p(t)\, dv(t)$ exists and

$$\left| \int_c^d p(x)\, dv(x) \right| \le \sup_{x \in [c,d]} |p(x)| \bigvee_c^d (v).$$

Utilising the representation (2.1) we have

$$\begin{aligned} \left| f(x)[u(b) - u(a)] - \int_a^b f(t)\, du(t) \right| &= \left| \int_a^x (x-t) \left(\int_0^1 f'[\lambda t + (1-\lambda) x]\, d\lambda \right) du(t) \right. \\ &\quad \left. + \int_x^b (x-t) \left(\int_0^1 f'[\lambda t + (1-\lambda) x]\, d\lambda \right) du(t) \right| \\ &\le \left| \int_a^x (x-t) \left(\int_0^1 f'[\lambda t + (1-\lambda) x]\, d\lambda \right) du(t) \right| \\ &\quad + \left| \int_x^b (x-t) \left(\int_0^1 f'[\lambda t + (1-\lambda) x]\, d\lambda \right) du(t) \right| \\ &\le \bigvee_a^x (u) \sup_{t \in [a,x]} [(x-t)\, \mu(f; x, t)] \\ &\quad + \bigvee_x^b (u) \sup_{t \in [x,b]} [(t-x)\, \mu(f; x, t)], \end{aligned}$$

and the theorem is proved. ■

Remark 1. Using the notations in Theorem 3, we have

$$\begin{aligned} M_1(x) &\le (x-a) \bigvee_a^x (u) \sup_{t \in [a,x]} \mu(f; x, t) \\ &\le (x-a) \bigvee_a^x (u) \cdot \begin{cases} \|f'\|_{[a,x],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \|f'\|_{[a,x],p} & \text{if } f' \in L_p[a,b],\ p \ge 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} M_2(x) &\le (b-x) \bigvee_x^b (u) \sup_{t \in [x,b]} \mu(f; x, t) \\ &\le (b-x) \bigvee_x^b (u) \cdot \begin{cases} \|f'\|_{[x,b],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \|f'\|_{[x,b],p} & \text{if } f' \in L_p[a,b],\ p \ge 1, \end{cases} \end{aligned}$$

for any $x \in [a, b]$.

The above inequalities for M_1 and M_2 are obvious from the inequality (2.4) and the details are omitted.

Remark 2. If we denote by $\|f'\|_{[c,d],p}$ the p norm on the interval $[c, d]$, where $1 \le p \le \infty$, then for $f' \in L_p[a, b]$, we have

$$\left| f(x)[u(b) - u(a)] - \int_a^b f(t)\, du(t) \right| \le (x-a) \bigvee_a^x (u) \|f'\|_{[a,x],p} + (b-x) \bigvee_x^b (u) \|f'\|_{[x,b],p} =: N(x), \tag{2.7}$$

where $p \in [1, \infty]$ and $x \in [a, b]$.

Obviously one can derive many upper bounds for the function $N(x)$ defined above. We intend to present in the following only a few that are simple and perhaps of interest for applications.

Estimate 1. We have

$$N(x) \le \left[(x-a)\bigvee_a^x(u) + (b-x)\bigvee_x^b(u)\right]\|f'\|_{[a,b],p}$$

$$\le \|f'\|_{[a,b],p}\cdot\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u); \\ \left[(x-a)^\alpha+(b-x)^\alpha\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}} & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\end{cases} \tag{2.8}$$

for any $x \in [a,b]$.

Estimate 2. We have

$$N(x) \le \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\bigvee_a^x(u)\|f'\|_{[a,x],p}+\bigvee_x^b(u)\|f'\|_{[x,b],p}\right]$$

$$\le \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]$$

$$\times\begin{cases}\max\left\{\|f'\|_{[a,x],p},\|f'\|_{[x,b],p}\right\}\bigvee_a^b(u); \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^x(u)\right)^q+\left(\bigvee_x^b(u)\right)^q\right]^{\frac{1}{q}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right]\end{cases}$$

for any $x \in [a,b]$.

Estimate 3. We have

$$N(x) \le \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[(x-a)\left\|f'\right\|_{[a,x],p}+(b-x)\left\|f'\right\|_{[x,b],p}\right]$$

$$\le \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]$$

$$\times\begin{cases}\max\left\{\|f'\|_{[a,x],p},\|f'\|_{[x,b],p}\right\}(b-a); \\ \left[(x-a)^q+(b-x)^q\right]^{\frac{1}{q}}\|f'\|_{[a,b],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right]\end{cases}$$

for each $x \in [a,b]$.

In practical applications, the mid-point rule, that results for $x = \frac{a+b}{2}$, is of obvious interest due to its simpler form.

Corollary 1. *With the assumptions in Theorem 3, we have the inequalities:*

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \le \frac{1}{2}(b-a)\left[\bigvee_a^{\frac{a+b}{2}}(u)\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\bigvee_{\frac{a+b}{2}}^b(u)\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],p}\right]$$

$$\leq \frac{1}{2}(b-a)\begin{cases} \max\left\{\|f'\|_{\left[a,\frac{a+b}{2}\right],p}, \|f'\|_{\left[\frac{a+b}{2},b\right],p}\right\}\bigvee_a^b(u); \\ \left[\|f'\|^{\alpha}_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|^{\alpha}_{\left[\frac{a+b}{2},b\right],p}\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{\beta}+\left(\bigvee_{\frac{a+b}{2}}^b(u)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ \quad \text{if } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^b(u)\right|\right]\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\right], \end{cases} \tag{2.9}$$

where $p\in[1,\infty]$.

From the above, it is obvious that we can get some appealing inequalities as follows:

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,\mathrm{d}u(t)\right|$$
$$\leq \frac{1}{2}(b-a)\begin{cases} \|f'\|_{[a,b],\infty}\bigvee_a^b(u), \quad \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^q+\left(\bigvee_{\frac{a+b}{2}}^b(u)\right)^q\right]^{\frac{1}{q}} \quad \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b]; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^b(u)\right|\right]\|f'\|_{[a,b],1}. \end{cases} \tag{2.10}$$

Remark 3. Similar inequalities can be obtained for the generalised trapezoid rule. We only state here the following simple result:

$$\left|\int_a^b u(t)\,\mathrm{d}f(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right|$$
$$\leq \frac{1}{2}(b-a)\begin{cases} \|f'\|_{[a,b],\infty}\bigvee_a^b(u), \quad \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^q+\left(\bigvee_{\frac{a+b}{2}}^b(u)\right)^q\right]^{\frac{1}{q}} \quad \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1, f'\in L_p[a,b]; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^b(u)\right|\right]\|f'\|_{[a,b],1} \end{cases}$$

provided that u is of bounded variation and f is absolutely continuous on $[a,b]$.

3. Bounds in the case of $|f'|$ a convex function

Some of the above results can be improved provided that a convexity assumption for $|f'|$ is in place:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$, $u:[a,b]\to\mathbb{R}$ a function of bounded variation on $[a,b]$ and $x\in[a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$ (and the intervals can be reduced to a single* *point), then*

$$
\begin{aligned}
&\left|[u(b)-u(a)] f(x)-\int_a^b f(t)\,\mathrm{d}u(t)\right| \\
&\le \frac{1}{2}\left[\bigvee_a^x(u) \sup_{t\in[a,x]}\left\{(x-t)\left|f'(t)\right|\right\}+\bigvee_x^b(u) \sup_{t\in[x,b]}\left\{(t-x)\left|f'(t)\right|\right\}\right] \\
&\quad +\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right] \\
&\le \frac{1}{2}\left[(x-a)\bigvee_a^x(u)\left\|f'\right\|_{[a,x],\infty}+(b-x)\bigvee_x^b(u)\left\|f'\right\|_{[x,b],\infty}\right] \\
&\quad +\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right], \qquad (3.1)
\end{aligned}
$$

for any $x \in [a, b]$.

Proof. As in the proof of Theorem 3, we have

$$
\begin{aligned}
&\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,\mathrm{d}u(t)\right| \le \sup_{t\in[a,x]}\left[(x-t)\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,\mathrm{d}\lambda\right|\right]\bigvee_a^x(u) \\
&\quad +\sup_{t\in[x,b]}\left[(t-x)\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,\mathrm{d}\lambda\right|\right]\bigvee_x^b(u) \\
&\le \sup_{t\in[a,x]}\left[(x-t)\int_0^1\left|f'[\lambda t+(1-\lambda)x]\right|\mathrm{d}\lambda\right]\bigvee_a^x(u) \\
&\quad +\sup_{t\in[x,b]}\left[(t-x)\int_0^1\left|f'[\lambda t+(1-\lambda)x]\right|\mathrm{d}\lambda\right]\bigvee_x^b(u) \\
&\le \sup_{t\in[a,x]}\left[(x-t)\frac{\left|f'(t)\right|+\left|f'(x)\right|}{2}\right]\bigvee_a^x(u)+\sup_{t\in[x,b]}\left[(t-x)\frac{\left|f'(t)\right|+\left|f'(x)\right|}{2}\right]\bigvee_x^b(u) \\
&\le \frac{1}{2}\left[\sup_{t\in[a,x]}\left\{(x-t)\left|f'(t)\right|\right\}\cdot\bigvee_a^x(u)+\sup_{t\in[x,b]}\left\{(t-x)\left|f'(t)\right|\right\}\cdot\bigvee_x^b(u)\right] \\
&\quad +\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right],
\end{aligned}
$$

which proves the first inequality in (3.1).

The second inequality in (3.1) is obvious using properties of sup and the theorem is completely proved. ■

The mid-point inequality is of interest in applications and provides a much simpler inequality:

Corollary 2. *If* f *and* u *are as above and* $|f'|$ *is convex on* $\left[a, \frac{a+b}{2}\right]$ *and* $\left[\frac{a+b}{2}, b\right]$*, then*

$$
\begin{aligned}
&\left|[u(b)-u(a)] f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,\mathrm{d}u(t)\right| \\
&\le \frac{1}{4}(b-a)\left[\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u)\\
&\le \frac{1}{4}(b-a)\bigvee_a^b(u)\left[\|f'\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \qquad (3.2)
\end{aligned}
$$

Remark 4. If we denote, from the second inequality in (3.1),

$$L_1(x) := \frac{1}{2}\left[(x-a)\|f'\|_{[a,x],\infty}\bigvee_a^x(u)+(b-x)\|f'\|_{[x,b],\infty}\bigvee_x^b(u)\right]$$

and

$$L_2(x) := \frac{1}{2}|f'(x)|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

for $x \in [a,b]$, then we can point out various upper bounds for the functions L_1 and L_2 on $[a,b]$.

For instance, we have

$$L_1(x) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

and by (3.1) we can state the following inequality of interest:

$$
\begin{aligned}
&\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,\mathrm{d}u(t)\right| \le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]\\
&\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases} \qquad (3.3)
\end{aligned}
$$

for each $x \in [a,b]$.

Remark 5. A similar result to (3.3) can be stated for the generalised trapezoid rule, out of which we would like to note the following one that is of particular interest:

$$
\begin{aligned}
&\left|\int_a^b u(t)\,\mathrm{d}f(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right|\\
&\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]\\
&\le \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases} \qquad (3.4)
\end{aligned}
$$

for each $x \in [a,b]$.

As in Corollary 2, the case $x = \frac{a+b}{2}$ in (3.4) provides the simple result

$$\left|\int_a^b u(t)\,\mathrm{d}f(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right|$$

$$\leq \frac{1}{4}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_{a}^{\frac{a+b}{2}}(u)+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^{b}(u)\right]+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_{a}^{b}(u)$$

$$\leq \frac{1}{4}(b-a)\bigvee_{a}^{b}(u)\left[\|f'\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{3.5}$$

Remark 6. Similar inequalities may be stated if one assumes either that $|f'|$ is quasi-convex or that $|f'|$ is log-convex on $[a,x]$ and $[x,b]$. The details are left to the interested readers.

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